Appendix C

LOCAL LINEARIZATION FOR THE EULER EQUATIONS

C.1 Derivation of the Flux Jacobians

In two dimensions, the Euler equations can be written as

$$\frac{\partial Q}{\partial t} + \frac{\partial \mathcal{E}}{\partial x} + \frac{\partial \mathcal{F}}{\partial y} = 0 \tag{C.1}$$

In eq. 2.6, the elements of the flux vector are written in terms of the *primitive* variables, ρ , u , v , e , and p. One can also write them in terms of the *conservative* variables, q_1, q_2, q_3 , and q_4 defined as

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix} \tag{C.2}$$

Thus

$$\mathcal{E} = \begin{bmatrix} q_2 \\ (\gamma - 1)q_4 + \frac{3-\gamma}{2}\frac{q_2^2}{q_1} - \frac{\gamma-1}{2}\frac{q_3^2}{q_1} \\ \frac{q_3 q_2}{q_1} \\ \gamma\frac{q_4 q_2}{q_1} - \frac{\gamma-1}{2}\left(\frac{q_2^3}{q_1^2} + \frac{q_3^2 q_2}{q_1^2}\right) \end{bmatrix} \tag{C.3}$$

$$\mathcal{F} = \begin{bmatrix} q_3 \\ \frac{q_3 q_2}{q_1} \\ (\gamma-1)q_4 + \frac{3-\gamma}{2}\frac{q_3^2}{q_1} - \frac{\gamma-1}{2}\frac{q_2^2}{q_1} \\ \gamma\frac{q_4 q_3}{q_1} - \frac{\gamma-1}{2}\left(\frac{q_2^2 q_3}{q_1^2} + \frac{q_3^3}{q_1^2}\right) \end{bmatrix} \tag{C.4}$$

From this it follows that the flux Jacobian of $\mathcal{E}$ written in terms of the conservative variables is

$$A = \frac{\partial \mathcal{E}_i}{\partial q_j} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & (3-\gamma)\left(\frac{q_2}{q_1}\right) & (1-\gamma)\left(\frac{q_3}{q_1}\right) & \gamma-1 \\ -\left(\frac{q_2}{q_1}\right)\left(\frac{q_3}{q_1}\right) & \left(\frac{q_3}{q_1}\right) & \left(\frac{q_2}{q_1}\right) & 0 \\ a_{41} & a_{42} & a_{43} & \gamma\left(\frac{q_2}{q_1}\right) \end{bmatrix} \tag{C.5}$$

where

$$\begin{aligned} a_{21} &= \frac{\gamma-1}{2}\left(\frac{q_3}{q_1}\right)^2 - \frac{3-\gamma}{2}\left(\frac{q_2}{q_1}\right)^2 \\ a_{41} &= (\gamma-1)\left[\left(\frac{q_2}{q_1}\right)^3 + \left(\frac{q_3}{q_1}\right)^2\left(\frac{q_2}{q_1}\right)\right] - \gamma\left(\frac{q_4}{q_1}\right)\left(\frac{q_2}{q_1}\right) \\ a_{42} &= \gamma\left(\frac{q_4}{q_1}\right) - \frac{\gamma-1}{2}\left[3\left(\frac{q_2}{q_1}\right)^2 - \left(\frac{q_3}{q_1}\right)^2\right] \\ a_{43} &= -(\gamma-1)\left(\frac{q_2}{q_1}\right)\left(\frac{q_3}{q_1}\right) \end{aligned} \tag{C.6}$$

and in terms of the primitive variables as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & (3-\gamma)u & (1-\gamma)v & (\gamma-1) \\ -uv & v & u & 0 \\ a_{41} & a_{42} & a_{43} & \gamma u \end{bmatrix} \tag{C.7}$$

where

$$a_{21} = \frac{\gamma-1}{2}v^2 - \frac{3-\gamma}{2}u^2$$

$$
\begin{aligned}
a_{41} &= (\gamma-1)u(u^2+v^2) - \gamma\frac{ue}{\rho} \\
a_{42} &= \gamma\frac{e}{\rho} - \frac{\gamma-1}{2}(3u^2+v^2) \\
a_{43} &= (1-\gamma)uv
\end{aligned}
\tag{C.8}
$$

Derivation of the two forms of $B = \partial\mathcal{F}/\partial Q$ is so similar that it is left as an exercise for the reader.

C.2 The Homogeneous Property of the Euler Equations

The Euler equations have a special property that is sometimes useful in constructing numerical methods. In order to examine this property, let us first inspect Euler's theorem on homogeneous functions. Consider first the scalar case. If $F(u, v)$ satisfies the identity

$$F(\alpha u, \alpha v) = \alpha^n F(u, v) \tag{C.9}$$

for a fixed n, F is called homogeneous of degree n. Differentiating both sides with respect to α and setting $\alpha = 1$ (since the identity holds for all α), we find

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} = nF(u, v) \tag{C.10}$$

Consider next the theorem as it applies to systems of equations. If the vector $F(Q)$ satisfies the identity

$$F(\alpha Q) = \alpha^n F(Q) \tag{C.11}$$

for a fixed n, F is said to be homogeneous of degree n and we find

$$\left[\frac{\partial F}{\partial q}\right] Q = nF(Q) \tag{C.12}$$

Now it is easy to show, by direct use of eq. C.11, that both $\mathcal{E}$ and $\mathcal{F}$ in eqs. C.3 and C.4 are homogeneous of degree 1, and their Jacobians, A and B, are homogeneous of degree 0 (actually the latter is a direct consequence of the former). This being the case, we notice that the expansion of the flux vector in the vicinity of t_n which, according to eq. 6.93 can be written in general as,

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_n + A_n(Q - Q_n) + O(h^2) \\ \mathcal{F} &= \mathcal{F}_n + B_n(Q - Q_n) + O(h^2) \end{aligned} \tag{C.13}$$

can be written

$$\begin{aligned} \mathcal{E} = A_n Q + O(h^2) \\ \mathcal{F} = B_n Q + O(h^2) \end{aligned} \tag{C.14}$$

since the terms $\mathcal{E}_n - A_n Q_n$ and $\mathcal{F}_n - B_n Q_n$ are identically zero for homogeneous vectors of degree 1, see eq. C.12. Notice also that, under this condition, the constant term drops out of eq. 6.94.

As a final remark, we notice from the chain rule that for *any* vectors F and Q

$$\frac{\partial F(Q)}{\partial x} = \left[\frac{\partial F}{\partial Q}\right]\frac{\partial Q}{\partial x} = A\frac{\partial Q}{\partial x} \tag{C.15}$$

We notice also that for a homogeneous F of degree 1, $F = AQ$ and

$$\frac{\partial F}{\partial x} = A\frac{\partial Q}{\partial x} + \left[\frac{\partial A}{\partial x}\right]Q \tag{C.16}$$

Therefore, if F is homogeneous of degree 1,

$$\left[\frac{\partial A}{\partial x}\right]Q = 0 \tag{C.17}$$

in spite of the fact that individually $[\partial A/\partial x]$ and Q are not equal to zero.